Derivation of Faulhbar's Formula

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Inspiration

- $$1 + 2 + 3 + ... + n = n(n+1)/2$$
- $$1^2 + 2^2 + 3^2 + ... + n^2 = n(n+1)(2n+1)/6$$
- $$1^3 + 2^3 + 3^3 + ... + n^3 = n^2(n+1)^2/4$$
- $$1^p + 2^p + 3^p + ... + n^p = ?$$

Derivation

- Let

$$G(x) = 1 + e^x + e^{2x} + e^{3x} + \ldots e^{nx} = \frac{e^{(n+1)x} - 1}{e^x - 1} \quad x \neq 0$$

- Note

$$G'0 = 1 + 2 + 3 \ldots n$$

$G^{(2)}(0) = 1^2 + 2^2 + 3^2 \ldots + n^2$
$G^{(3)}(0) = 1^3 + 2^3 + 3^3 \ldots + n^3$
$G^{(p)}(0) = 1^p + 2^p + 3^p \ldots + n^p$

Derivation

Let

$$G'(0) = S_1(n) = 1 + 2 + 3 + ... + n$$

$$G''(0) = S_2(n) = 1^2 + 2^2 + 3^2 + ... + n^2$$

$$G'''(0) = S_3(n) = 1^3 + 2^3 + 3^3 + ... + n^3$$

$$G^{(p)}(0) = S_p(n) = 1^p + 2^p + 3^p + ... + n^p$$

Using Taylor series expansion of G(x), we get

$$G(x) = S_0(n) + S_1(n) \cdot \frac{x}{1!} + S_2(n) \cdot \frac{x^2}{2!} + S_3(n) \cdot \frac{x^3}{3!} + ... + S_p(n) \cdot \frac{x^p}{p!} ...$$

Where

$$S_p(n) = 1^p + 2^p + 3^p + ... + n^p$$

Derivation

Remember

$$G(x) = \frac{x}{e^x - 1} \cdot \frac{e^{Nx-1}}{x} \text{ where N} = \text{n} + 1 \text{ and } x \neq 0$$

Note, now we have the product of two sequences:

$$\frac{x}{e^x - 1} = B_0 + B_1 \frac{x}{1!} + B_2 \frac{x^2}{2!} + B_3 \frac{x^3}{3!} \ldots$$

$$\frac{e^{Nx-1}}{x} = N + N^2 \frac{x}{2!} + N^3 \frac{x^2}{3!} + N^4 \frac{x^3}{4!} \ldots$$

Derivation

We want to combine these two products

$$\frac{x}{e^x - 1} = B_0 + B_1\frac{x}{1!} + B_2\frac{x^2}{2!} + B_3\frac{x^3}{3!}\ldots$$

$$\frac{e^{Nx-1}}{x} = N + N^2\frac{x}{2!} + N^3\frac{x^2}{3!} + N^4\frac{x^3}{4!}\ldots$$

$$(B_0 + B_1\frac{x}{1!} + B_2\frac{x^2}{2!} + B_3\frac{x^3}{3!}\ldots) \cdot (N + N^2\frac{x}{2!} + N^3\frac{x^2}{3!} + N^4\frac{x^3}{4!}\ldots)$$

We want to match all the terms in the form $\frac{x^i}{i!}$ so we can use...

Theorem (Cauchy Product of Power Series)

$$(\sum_{k=0}^{\infty} a_k \cdot \frac{x^k}{k!}) \cdot (\sum_{m=0}^{\infty} b_m \cdot \frac{x^m}{m!}) = \sum_{k=0}^{\infty}\sum_{m=0}^{k} \binom{k}{m} a_m \cdot b_{k-m} \cdot \frac{x^k}{k!}$$

Derivation

Note

$$N + N^2\frac{x}{2!} + N^3\frac{x^2}{3!} + N^4\frac{x^3}{4!}\ldots = \frac{1}{x}\sum_{m=1}^{\infty} N^m \cdot \frac{x^m}{m!}$$

So our product

$$(B_0 + B_1\frac{x}{1!} + B_2\frac{x^2}{2!} + B_3\frac{x^3}{3!}\ldots)\cdot(N + N^2\frac{x}{2!} + N^3\frac{x^2}{3!} + N^4\frac{x^3}{4!}\ldots)$$

is the same as

$$(\sum_{k=0}^{\infty} B_k \cdot \frac{x^k}{k!})\cdot(\sum_{m=1}^{\infty} N^m \cdot \frac{x^m}{m!})\cdot\frac{1}{x}$$

Derivation

We still can't apply our theorem (Cauchy Product of Power Series) because

$$\frac{1}{x}(\sum_{k=0}^{\infty} B_k \frac{x^k}{k!})(\sum_{m=1}^{\infty} N^m \frac{x^m}{m!})$$

has the bounds m = 1 to infinity. We want it to start at m = 0. Note, our series is

$$\frac{1}{x}(B_0 + B_1 \frac{x}{1!} + B_2 \frac{x^2}{2!} + B_3 \frac{x^3}{3!} ...)(N + N^2 \frac{x^2}{2!} + N^3 \frac{x^3}{3!} + N^4 \frac{x^4}{4!} ...)$$

Collecting like terms for x...

$$\frac{1}{x}((B_0 \frac{x^0}{0!} N^1 \frac{x^1}{1!}) + (B_0 \frac{x^0}{0!} N^2 \frac{x^2}{2!} + B_1 \frac{x^1}{1!} N \frac{x^1}{1!}) + ...)$$

Derivation

$$\frac{1}{x}(\sum_{k=0}^{\infty} B_k \frac{x^k}{k!})(\sum_{m=1}^{\infty} N^m \frac{x^m}{m!})$$

$$= \frac{1}{x}((B_0 \frac{x^0}{0!} N^1 \frac{x^1}{1!}) + (B_0 \frac{x^0}{0!} N^2 \frac{x^2}{2!} + B_1 \frac{x^1}{1!} N \frac{x^1}{1!}) + \ldots)$$

Note the term

$$(B_k \frac{x^k}{k!} \cdot N^0 \frac{x^0}{0!})$$

never appears, thus

$$\frac{1}{x} \sum_{k=1}^{\infty} \sum_{m=0}^{k-1} \binom{k}{m} B_m N^{k-m} \frac{x^k}{k!}$$

Derivation

$$\frac{1}{x}\sum_{k=1}^{\infty}\sum_{m=0}^{k-1}\binom{k}{m}B_m N^{k-m}\frac{x^k}{k!}$$

let n = k - 1

$$\frac{1}{x}(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n+1}{m}B_m N^{n+1-m}\frac{x^{n+1}}{(n+1)!})$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n+1}{m}\frac{B_m N^{n+1-m}}{n+1}\frac{x^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n+1}{m}\frac{B_m N^{n+1-m}}{n+1}\frac{x^n}{n!}$$

Derivation

As of now, we have

$$S_p(n) = G^{(p)}(0) = \frac{1}{p+1} \sum_{p=0}^{\infty} \sum_{m=0}^{p} \binom{p+1}{m} B_m N^{p+1-m} \frac{x^p}{p!}$$

Note

$$S_p(N) = S_p(n) + N^p = \frac{1}{p+1} \sum_{m=0}^{p} \binom{p+1}{m} B_m N^{p+1-m} + N^p$$

$$= \frac{1}{p+1}(\binom{p+1}{0} B_0 N^{p+1} + \binom{p+1}{1} B_1 N^P + \binom{p+2}{2} B_2 N^{p-1} + \ldots)$$

$$+N^P$$

Derivation

$$S_p(N) = \frac{1}{p+1}(\binom{p+1}{0}B_0N^{p+1} + \binom{p+1}{1}B_1N^P +$$
$$\binom{p+2}{2}B_2N^{p-1} + ...) + N^P$$

Note

$$B_i = 0 \quad \forall \text{ odd } i > 1$$

and

$$B_1 = -\frac{1}{2} = -1 - B_1$$

Combining the N^p term with $\frac{1}{p+1} \cdot \binom{p+1}{1}B_1N^p$, we get $-N^P - B_1N^P + N^P = -B_1N^P$. Thus we can write,

$$S_p(N) = \frac{1}{p+1}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j}B_jN^{p+1-j}$$

Conclusion

A closed form for $1^p + 2^p + 3^p ... + n^p$

is

$$\frac{1}{p+1}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j}B_jN^{p+1-j}$$

Thank You

Thank you for listening!

Questions?